

The University of Chicago

AN ANALOGUE OF GREEN'S
THEOREM FOR MULTIPLE INTEGRAL
PROBLEMS IN THE CALCULUS
OF VARIATIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1941

By

ALBERT B. CARSON

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CHICAGO, ILLINOIS

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AN ANALOGUE OF GREEN'S THEOREM FOR MULTIPLE INTEGRAL
PROBLEMS IN THE CALCULUS OF VARIATIONS

1. Introduction. The problem to be considered is that of finding necessary and sufficient conditions for the following equation to hold

$$(1.1) \int_A [\bar{v}_\alpha(x) \eta_{x_\alpha}(x) + w(x) \eta(x)] dA = \int_B \eta(x) v_\alpha(x) \lambda_\alpha dB, \\ (\alpha = 1, \dots, m)$$

where the integral on the left is taken over a bounded region A of points $(x) = (x_1, \dots, x_m)$ and that on the right is taken over the boundary B of A . The symbol λ_α represents the direction cosines of the outer normal to B . By the value of an integral of the form $\int f(x) dB$ over a piece of the boundary B defined by functions $x_1(t)$, (t on T), will be meant the integral $\int_T f(x(t)) \sqrt{c_\alpha c_\alpha} dt_1, \dots, dt_{m-1}$, where c_α is the cofactor of a_α in the determinant $|a_\alpha \frac{\partial x_\alpha}{\partial t_\nu}|$, ($\nu = 1, \dots, m-1$). The method used in this paper is essentially a generalization and extension of that used by Hestenes [XII]* in the two-dimensional case. The proofs here given have the advantage that no assumptions need to be made on the behavior of the functions v_α outside of the region $A + B$. Moreover, no assumptions need to be made regarding the intersections of the boundary with hyperplanes parallel to the coordinate hyperplanes.

Among other things it is shown that equation (1.1) holds if and only if the relation

*Roman numerals in brackets refer to the bibliography at the end of this paper.

$$(1.2) \quad \int_{A'} w(x) dA = \int_{B'} v_{\alpha}(x) \chi_{\alpha} dB$$

is satisfied for every m -dimensional cube A' whose closure is in A and whose sides are parallel to the coordinate hyperplanes. This is done under very general hypotheses concerning the boundary B and the functions $v_{\alpha}(x)$ and $w(x)$. If the function $w(x)$ is continuous and the functions $v_{\alpha}(x)$ have continuous derivatives then equation (1.2) is equivalent to the equation $w = v_{\alpha} x_{\alpha}$. Setting $\eta = 1$ and $w = v_{\alpha} x_{\alpha}$ in equation (1.1), one obtains Green's formula

$$\int_A v_{\alpha} x_{\alpha} dA = \int_B v_{\alpha} \chi_{\alpha} dB.$$

Thus equation (1.1) can be considered to be an analogue of Green's formula. Since our proofs are made independently of Green's theorem, they constitute a proof of this theorem for m -dimensional space.

Equation (1.1) arises in the study of the first variation of a multiple integral of the form

$$\int_A f[x_1, \dots, x_m, y_1(x), \dots, y_n(x), \frac{\partial y_1}{\partial x_1}, \dots, \frac{\partial y_n}{\partial x_m}] dA.$$

By means of this equation and, in particular, equation (1.2), one obtains the Haar-Coral equations

$$\int_A f_{y_i} dA = \int_B f_{y_i x_{\alpha}} \chi_{\alpha} dB, \quad (i = 1, \dots, n)$$

edge conditions, and an analogue of Jacobi's condition for a minimizing manifold of class c' .

After formulating the problem in Section 2, we proceed in Section 3 to show that equation (1.1) holds for all admissible functions if and only if it holds for certain special classes of admissible functions. Section 4 is devoted to the proof of Green's Theorem and other associated results. The previous

results are generalized further in Sections 5 and 6 by considering a more general type of boundary B and discussing the case in which $v_\alpha(x)$ have certain types of discontinuities. In Section 8, some of the previous results are applied in proving theorems which are useful in mechanics. The remainder of the paper is concerned with applications to the problem of the calculus of variations for multiple integrals and includes the derivation of Haar-Coral equations, edge conditions, and an analogue of Jacobi's condition.

2. Analytic formulation of the problem. In Sections 3 and 4 a region A of points $(x) \equiv (x_1, \dots, x_m)$ will be called an admissible region in case it is bounded and its boundary B is non-singular and of class c' . Then the boundary B can be represented locally by equations of the form $x_\alpha = x_\alpha(t_1, \dots, t_{m-1})$, ($\alpha = 1, \dots, m$), where the functions $x_\alpha(t)$ are of class c' and the matrix $\left\| \frac{\partial x_\alpha}{\partial t_\gamma} \right\|$, ($\gamma = 1, \dots, m-1$), is of rank $m-1$. A function will be said to be of class c' on $A+B$ in case it is continuous and has a continuous differential on $A+B$.

By an admissible function $\eta(x)$ will be meant one belonging to a class $\mathcal{H}$ of continuous functions on $A+B$ having the following properties: (1) Every non-negative function of class c^∞ on $A+B$ is in $\mathcal{H}$; (2) The product of a function in $\mathcal{H}$ by a non-negative function of class c^∞ on $A+B$ is in $\mathcal{H}$; (3) On each interval $a_\alpha \leq x_\alpha \leq b_\alpha$ in A , a function $\eta(x)$ in $\mathcal{H}$ is absolutely continuous in each variable $x_1, \dots, x_m$ when the remaining variables are fixed and the derivatives $\pi_\alpha \equiv \eta_{x_\alpha}$ are integrable on A .

The smallest class of admissible functions is the class H_1 of all non-negative functions of class c^∞ on $A+B$. The largest class is the class H_2 of all functions having property (3). Any subclass of H_2 containing H_1 and having property (2) is admissible. The class commonly used in the calculus of variations

is the class of all functions of class c' on $A + B$. These functions form a class of admissible functions.

Let $v_1(x), \dots, v_m(x)$ be m real valued functions which are continuous on $A + B$, and let $w(x)$ denote a real single valued function that is integrable on A in the sense of Lebesgue.

We shall find necessary and sufficient conditions for the following equation to hold

$$(2.1) \quad \int_A (v_\alpha \pi_\alpha + w \eta) dA = \int_B \eta v_\alpha \chi_\alpha dB,$$

where $\chi_1, \dots, \chi_m$ are the direction cosines of the outer normal to B . Here and elsewhere the repetition of a subscript will denote summation unless otherwise specified.

In Section 5, the class of admissible regions will be enlarged and in Section 6, we shall discuss the case where the functions v_α have certain types of discontinuities. We restrict ourselves to the case here described in order to simplify the argument.

In Section 3, the following result will be used.

Lemma 2.1. The form of equation (2.1) is unaltered by a rotation or translation of axes.

Since the form of equation (2.1) is obviously unchanged by a translation, consider a rotation $x_\alpha = A_{\alpha\beta} z_\beta$, ($\alpha, \beta = 1, \dots, m$). Let $y(z)$ represent the function into which $\eta(x)$ is transformed by this rotation. The derivatives $p_\alpha \equiv y_{z_\alpha}$ and $\pi_\alpha \equiv \eta_{x_\alpha}$ are connected by the formula $\pi_\alpha = A_{\alpha\beta} p_\beta$. If we set $V_\beta \equiv v_\alpha A_{\alpha\beta}$ then $V_\beta p_\beta = v_\alpha \pi_\alpha$. Moreover, the direction cosines of the normal to B are related by the formula $\chi_\alpha = A_{\alpha\beta} L_\beta$; where the symbol L_β represents the new direction cosines of the normal to the boundary B . Thus, we see that $v_\alpha \chi_\alpha = v_\alpha (A_{\alpha\beta} L_\beta) = V_\beta L_\beta$ and integral (1.1) takes the form

$$\int_A (v_\beta p_\beta + wy) dA = \int_B y v_\beta L_\beta dB.$$

This completes the proof of Lemma 2.1.

3. Preliminary Theorems. In this section it will be shown that equation (2.1) holds for all admissible functions $\eta(x)$ if and only if it holds for certain special classes of admissible functions. As a first result, we have

Theorem 3.1. The equation (2.1) holds for every admissible function $\eta(x)$ if and only if to each point P on $A + B$ there is a neighborhood N of P such that equation (2.1) is true for every admissible function $\eta(x)$ which is identically zero on $A - AN$.

In order to prove this, cover $A + B$ with a finite number of spheres $K_1, K_2, \dots, K_q$ with centers $P_1, \dots, P_q$ on $A + B$ and radii $r_1, \dots, r_q$ such that equation (2.1) holds for every admissible function $\eta(x)$ which vanishes on $A - AK_1'$, where K_1' is a sphere of radius $2r_1$ about the point P_1 . Let $d_1(x)$ be the distance from the point (x) to the point P_1 and set $h_1(x) = h(\frac{d_1(x)}{r_1})$, where $h(t)$ is defined to be a function of class C^∞ and such that for $t \leq 1$, $h(t) = 0$; for $t \geq 2$, $h(t) = 1$; and $h'(t) \geq 0$. A function of this type used by Hestenes [XII] in a similar situation is the function $h(t)$ having $h(t) = 0$ for $t \leq 1$; $h(t) = 1$ for $t \geq 2$; and $h(t) = e^{g(t)}/1-t$ for $1 < t < 2$, where $g(t) = e^{1/t-2}$.

Let $\eta(x)$ be any admissible function and consider the functions

$$\eta_1 = (1 - h_1)\eta,$$

$$\eta_2 = h_1(1 - h_2)\eta,$$

$$\dots$$

$$\eta_q = h_1 h_2 \dots h_{q-1} (1 - h_q) \eta.$$

(3.1)

Adding, we obtain $\eta = \eta_1 + \eta_2 + \dots + \eta_q + h_1 h_2 \dots h_q \eta$. Since the function h_1 vanishes on sphere K_1 and the spheres $K_1, \dots, K_q$ cover $A + B$, the product $h_1 h_2 \dots h_q$ is identically zero on $A + B$ and we have $\eta = \eta_1 + \eta_2 + \dots + \eta_q$ on $A + B$. Let

$$L(\eta) = \int_A (v \nabla \eta + w \eta) dA - \int_B \eta v_\alpha x_\alpha dB.$$

Since the functions η_1 vanish on $A - AK_1'$, it follows that $L(\eta_1) = 0$. Due to the linearity of $L(\eta)$, we have

$$\begin{aligned} (3.2) \quad L(\eta) &= L(\eta_1 + \eta_2 + \dots + \eta_q) \\ &= L(\eta_1) + L(\eta_2) + \dots + L(\eta_q) = 0 \end{aligned}$$

as was to be proved.

Theorem 3.2. Equation (2.1) holds for every admissible function $\eta(x)$ if and only if it holds for every admissible function $\eta(x)$ which vanishes on a neighborhood of the boundary.

Using the criterion that equation (2.1) holds for every admissible function $\eta(x)$ vanishing on a neighborhood of the boundary, we know that to each point P of A there is a neighborhood N of P such that $L(\eta) = 0$ for every admissible function $\eta(x)$ which vanishes on $A - AN$. In order to complete the proof by use of Theorem 3.1 we will show that points of B have the same property.

By Lemma 2.1, we can assume that the boundary point P under consideration is the origin and that the positive x_m -axis is the inner normal of B at P . We define a neighborhood N of P by the inequalities

$$(3.3) \quad -e < x_\gamma < e, \quad -e' < x_m < e' \quad (\gamma = 1, \dots, m-1),$$

where e, e' are so small that the boundary of B in N can be defined in this neighborhood by an equation of the form

$x_m = f(x_1, x_2, \dots, x_{m-1})$, the function f being single-valued and of class c' in the region defined by inequalities (3.3). Let

$$H(x, r) = h\left(\frac{x_m - f}{r}\right)$$

where r is an arbitrary positive parameter and $h(t)$ is the function used in proof of Theorem 3.1 above. We have $H = 0$ when $x_m \leq f + r$ and $H = 1$ when $x_m \geq f + 2r$. Consider an admissible function $\eta(x)$ vanishing on $A - AN$ and let $\mathcal{J}$ be equal to $H\eta$ on N and identically zero elsewhere. The function $\mathcal{J}$ is admissible and vanishes at all points of N for which $x_m \leq f + r$. Therefore, $\mathcal{J}$ vanishes in a neighborhood of the boundary B , so that $L(\mathcal{J}) = 0$ by virtue of our hypothesis. Thus we have

$$\begin{aligned} 0 = L(\mathcal{J}) &= \int_{AN} (v_\alpha \mathcal{J}_{x_\alpha} + w \mathcal{J}) dA - \int_{BN} \mathcal{J} v_\alpha \chi_\alpha dB \\ &= \int_{AN} H(v_\alpha \eta_{x_\alpha} + w \eta) dA + \int_{AN} \eta v_\alpha H_{x_\alpha} dA \end{aligned}$$

from which we find that

$$\int_{AN} H(v_\alpha \eta_{x_\alpha} + w \eta) dA = - \int_{AN} \eta v_\alpha H_{x_\alpha} dA.$$

The first member of this equation has the first member of equation (2.1) as its limit since H approaches 1 when r tends to zero. It will be shown that the remaining part of this equation has the second member of equation (2.1) as its limit. In this connection, we note that $H_{x_\alpha} = 0$ when $x_m \geq f + 2r$. Taking r sufficiently small and setting $x_m = f + rt$, we have

$$- \int_{AN} \eta(x) v_\alpha(x) H_{x_\alpha} dA = \int_{AN} \eta(X) v_\alpha(X) h'(t) \chi_\alpha \frac{\sqrt{1 + f_{x_\beta}^2 f_{x_\beta}^2}}{r} dA$$

where $X \equiv (x_1, \dots, x_{m-1}, f + rt)$, $(\beta = 1, \dots, m)$, and the symbol χ_α represents the direction cosines of the outer normal to the boundary B . The integral may then be written in the form

$$\int_{BN} dB \int_0^2 \eta(x) v_\alpha(x) \chi_\alpha h'(t) dt.$$

Taking the limit as r approaches zero, the result is

$$\int_{BN} \eta(x) v_\alpha(x) \chi_\alpha dB \int_0^2 h'(t) dt = \int_B \eta(x) v_\alpha(x) \chi_\alpha dB$$

since η vanishes outside BN . This completes the proof of Theorem 3.2.

Theorem 3.3. The equation (2.1) holds for every admissible function $\eta(x)$ if and only if to each point P of A there is a neighborhood N of P whose closure is in A and such that equation (2.1) holds for every admissible function $\eta(x)$ which vanishes on $A - AN$.

In proof of this result, let $\eta(x)$ be an admissible function vanishing on a neighborhood N of boundary B . Proceeding as in the proof of Theorem 3.1 with $A + B$ replaced by $A - AN$, cover the bounded closed set $A - AN$ by a finite number of spheres $K_1, \dots, K_q$ with centers $P_1, \dots, P_q$ on A and radii $r_1, \dots, r_q$. These spheres can be chosen such that for each i the sphere K_i' of radius $2r_i$ about point P_i lies interior to A and is such that equation (2.1) holds for every admissible function which vanishes on $A - AK_i$. Construct functions $h_i(x)$ related to the spheres K_i and K_i' , as in the proof of Theorem 3.1. Let $\eta_1, \dots, \eta_q$ be given by the formulas (3.1). Then $\eta = \eta_1 + \dots + \eta_q$ on $A - AN$. Since, however, η and hence $\eta_1, \dots, \eta_q$ are zero on AN it follows that $\eta = \eta_1 + \dots + \eta_q$ on $A + B$. By use of the formulas $L(\eta_i) = 0$, ($i = 1, \dots, q$), it is found that equation (3.2) holds. The criterion described in Theorem 3.2 is accordingly implied by the one given in Theorem 3.3. The converse is immediate. Theorem 3.3 now follows from Theorem 3.2.

Theorem 3.4. The equation (2.1) holds for every admissible function $\eta(x)$ if and only if it holds with A, B replaced by A_0 ,

B_0 for every admissible subregion A_0 of A whose boundary B_0 is also in A .

An admissible subregion A_0 of A is, of course, one whose boundary B_0 has the properties required for the boundary B of A . In this theorem the class of admissible functions associated with $A_0 + B_0$ is understood to be the class of admissible functions that is associated with the original set $A + B$. The result in Theorem 3.4 can be proved by use of Theorem 3.3. Let P be any point of A . There is a spherical neighborhood A_0 of P whose closure is in A . This region is obviously an admissible subregion of A . By virtue of the criterion of Theorem 3.4, equation (2.1) holds on subregion $A_0 + B_0$ for every admissible function $\eta(x)$ and, in particular, it holds if $\eta(x)$ vanishes on $A - A_0$. Thus the hypotheses of Theorem 3.3 are satisfied and application of this theorem completes the proof of Theorem 3.4.

Theorem 3.5. The results described in Theorems 3.1 to 3.4 hold when A is an interval represented by inequalities of the form $a_\alpha < x_\alpha < b_\alpha$, ($\alpha = 1, \dots, m$).

This result follows from the more general result given in Theorem 5.2. A more elementary proof, however, is given below. The arguments for an interval A are like those given previously except in the proof of Theorem 3.2, where an additional argument is needed in order to show that equation (2.1) holds when the criterion described in Theorem 3.2 is satisfied. In order to prove Theorem 3.2 for an interval A let us suppose that the formula (2.1) holds for every admissible function $\eta(x)$ that is identically zero on a neighborhood of k ($k \leq 2m$) of the faces of the interval A . We shall show that if k is greater than zero, then the formula (2.1) holds, also, for every admissible function having $\eta(x) \equiv 0$ on a neighborhood of $K - 1$ of the faces of A .

To this end, consider an admissible function $\eta(x)$ that is identically zero on a neighborhood of $k - 1$ faces of A . We can suppose that there is no neighborhood of the face $x_1 = a_1$ on which η is identically zero. For convenience, suppose $a_1 = 0$. Let $h(t)$ be the function used in the proof of Theorem 3.1 and set $H(x, r) = h(\frac{x_1}{r})$, where r is an arbitrary positive constant. Since $H = 0$ when $x_1 < r$, the variation $y = H\eta$ vanishes on a neighborhood of k of the faces of A and hence $L(y) = 0$, that is

$$(3.4) \quad \int_A (v_\alpha \Gamma_\alpha + w\eta) dA = - \int_A \eta v_\alpha H_{x_\alpha} dA + \int_{B_1} H \eta v_\alpha \lambda_\alpha dB,$$

where B_1 is the boundary of A with the face $x_1 = 0$ excluded.

This face can be excluded, since $H = 0$ when $x_1 = 0$. As r tends to zero, the first and last integrals have respectively the limits

$$(3.5) \quad \int_A (v_\alpha \Gamma_\alpha + w\eta) dA, \quad \int_{B_1} \eta v_\alpha \lambda_\alpha dB,$$

since $\lim_{r=0} H = 1$ on $A + B$ except on the face $x_1 = 0$. The remaining integral can be written in the form

$$- \int_{a_2}^{b_2} \dots \int_{a_m}^{b_m} dx_2, \dots, dx_m \int_0^{2r} \eta v_1 h'(\frac{x_1}{r}) \frac{dx_1}{r} \quad (2r < b_1)$$

since $H_{x_\alpha} = 0$ ($\alpha > 1$), $H_{x_1} = h'(\frac{x_1}{r})/r$ and $H_{x_1} = 0$ when $x_1 > 2r$.

Setting $x_1 = rt$ this integral becomes

$$- \int_{a_2}^{b_2} \dots \int_{a_m}^{b_m} dx_2, \dots, dx_m \int_0^2 \eta v h'(t) dt.$$

Letting r tend to zero and using the relations $h(0) = 0$, $h(2) = 1$, we obtain as a limit the expression

$$(3.6) \quad - \int_{a_2}^{b_2} \dots \int_{a_m}^{b_m} v_1 dx_2, \dots, dx_m = \int_{B_2} \eta v_\alpha \lambda_\alpha dB$$

where B_2 is the face $x_1 = 0$ and $\lambda_1, \lambda_2, \dots, \lambda_m$ are the direction cosines $(-1, 0, \dots, 0)$ of the outer normal of B_2 relative to A .

In view of equation (3.4) and the limits (3.5) and (3.6), it

follows that equation (2.1) holds for an admissible function $\eta(x)$ having $\eta(x) \equiv 0$ on a neighborhood of $k - 1$ of the faces of A . Letting k take successively the values $2m$, $2m - 1$, ..., 1 it is seen from the result just established that the criterion described in Theorem 3.2 implies that equation (2.1) holds for an interval A . This completes the proof of Theorem 3.5.

4. Analogue of Green's Theorem. If η is set identically equal to one in equation (2.1) this relation takes the form

$$(4.1) \quad \int_A w \, dA = \int_B v_\alpha \lambda_\alpha \, dB.$$

This result suggests the following criterion for equation (2.1) to hold.

Theorem 4.1. The equation (2.1) holds if and only if the formula

$$(4.2) \quad \int_{A'} w \, dA = \int_{B'} v_\alpha \lambda_\alpha \, dB$$

holds for every m -dimensional cube A' whose closure is in A and whose faces are parallel to the coordinate hyperplanes.

Let A' represent an m -dimensional cube on A at a distance $d > 0$ from the boundary B and whose faces are parallel to the coordinate hyperplanes. Each point of A is interior to a cube of this type. By Theorem 3.3, it will be sufficient to show that equation (2.1) holds for every admissible function $\eta(x)$ which vanishes on $A - AA'$ when the criterion described in Theorem 4.1 is satisfied. We shall make use of the following integral means*

$$(4.2) \quad V_\alpha(x, r) \equiv \frac{1}{(2r)^m} \int_{A_r} v_\alpha(s) dA; \quad W(x, r) \equiv \frac{1}{(2r)^m} \int_{A_r} w(s) dA, \\ (0 < r < d),$$

*For a discussion of integral means see, for example, Levin [XIV].

where $(s) \equiv (s_1, \dots, s_m)$ and A_r is the region $x_\alpha - r \leq s_\alpha \leq x_\alpha + r$, ($\alpha = 1, \dots, m$). By hypothesis, we have

$$\int_{A'} w \, dA - \int_{B'} v_\alpha \chi_\alpha \, dB = 0.$$

When we divide each member of this equation by $(2r)^m$ and use relations (4.2), we find that $W - V_\alpha x_\alpha = 0$ on $A' + B'$.

If $\eta(x)$ is any admissible function, we have, by means of iterated integration, the relation

$$\int_{A'} (v_\alpha \pi_\alpha + w\eta) dA = \int_{A'} \left[\frac{\partial}{\partial x_\alpha} (v_\alpha \eta) + \eta (W - v_\alpha x_\alpha) \right] dA = \int_{B'} \eta v_\alpha \chi_\alpha \, dB.$$

Thus, it is seen that

$$\int_{A'} (v_\alpha \pi_\alpha + w\eta) dA = \int_{B'} \eta v_\alpha \chi_\alpha \, dB.$$

Taking the limit as r tends to zero, the result is

$$\int_{A'} (v_\alpha \pi_\alpha + w\eta) dA = \int_{B'} \eta v_\alpha \chi_\alpha \, dB.$$

Thus, equation (2.1) holds for any admissible function $\eta(x)$ which vanishes on $A - AA'$. This result and the application of Theorem 3.3 complete the proof of the direct theorem. The proof of the converse follows immediately from Theorem 3.5.

Corollary 1. The equation (4.1) holds for A when it holds for every m-dimensional cube whose closure is in A and whose faces are parallel to the coordinate hyperplanes.

It is interesting to observe in passing, that the cubes $A' + B'$ used above are not admissible subregions in the sense described in Section 2, but Theorem 3.3 does not require that the neighborhoods N be admissible subregions of A .

As a consequence of Theorem 4.1, we have the following result.

Theorem 4.2. Suppose that on each cube $a_\alpha \leq x_\alpha \leq b_\alpha$ in A

the functions v_α are absolutely continuous in each variable x_β when the remaining variables are fixed, except on a set of $(m-1)$ -dimensional measure zero. Suppose, also, that $v_\alpha x_\alpha$ are integrable on A . Then equation (2.1) holds if and only if $w = v_\alpha x_\alpha$ almost everywhere on A .

By Theorem 4.1, equation (2.1) holds if and only if equation (4.1) holds for all m -dimensional cubes A' whose faces are parallel to the coordinate hyperplanes. It remains to be shown that equation (4.1) holds for such cubes if and only if $w = v_\alpha x_\alpha$ almost everywhere on A .

By iterated integration it can be shown that

$$\int_{A'} v_\alpha x_\alpha dA = \int_{B'} v_\alpha \lambda_\alpha dB$$

where λ_α are direction cosines of the outer normal to B' . Equation (4.1) holds if $w = v_\alpha x_\alpha$. Therefore, we have the relation

$$\int_{A'} w dA = \int_{B'} v_\alpha \lambda_\alpha dB = \int_{A'} v_\alpha x_\alpha dA.$$

From this follows the relation

$$\int_{A'} (w - v_\alpha x_\alpha) dA = 0$$

which holds for all cubes A' with sides parallel to the coordinate hyperplanes only if $w = v_\alpha x_\alpha$ almost everywhere on A . This completes the proof of Theorem 4.2.

We have as an immediate consequence of this result the following corollary which is discussed by Huke [XIII, pp. 55 ff.].

Corollary 1. The equation

$$\int_A w \eta dA = 0$$

holds for every admissible function $\eta(x)$ with $\eta = 0$ on B , if and only if $w = 0$ almost everywhere on A .

This result follows from Theorems 3.2 and 4.2 with $v_\alpha = 0$. When $w = 0$ we have by these theorems the following further result.

Corollary 2. If the functions $v_\alpha(x)$ are of class c' on A , the equation

$$(4.4) \quad \int_A v_\alpha \pi_\alpha dA = 0$$

holds for every admissible function η having $\eta = 0$ on B , if and only if $v_\alpha x_\alpha = 0$. Moreover, if $v_\alpha x_\alpha = 0$ on A one has

$$(4.5) \quad \int_{B'} v_\alpha x_\alpha dB = 0$$

where B' is the boundary of an admissible subregion of A .

Green's theorem may be obtained as an immediate consequence of Theorem 4.2. This constitutes a proof of the theorem for an m -dimensional region A , having a boundary B of the type described at the beginning of Section 2. The proof will be extended to hold for a region having a more general type of boundary in Section 5.

Theorem 4.3. (Green's theorem). Under the hypotheses of Theorem 4.2, the equation (4.1) holds when $w = v_\alpha x_\alpha$ almost everywhere on A .

By Theorem 4.2, equation (2.1) holds with $w = v_\alpha x_\alpha$. Setting $\eta = 1$ it is seen that equation (4.1) holds with $w = v_\alpha x_\alpha$, as was to be proved. This theorem is sometimes called Green's Lemma or Gauss's Theorem.

5. Extension of the class of admissible regions. In order to simplify the argument in the previous sections, we restricted ourselves to intervals and to regions having a boundary of class c' and non-singular. In this section, we shall consider a more general type of boundary as described in the following theorem.

Theorem 5.1. The results described in Sections 3 and 4

are valid if the class of admissible regions A is extended to include regions whose boundary B can be subdivided into a finite number of non-singular $(m - 1)$ -dimensional subspaces of class c' whose intersections form a set E having the following property: there exist positive numbers q and s with $s < m - 1$ such that, given a number $r > 0$, the quantity q/r^s is an upper bound of the number of points in a set D, on E, chosen so that the distance between any two points of D is at least r .

Let r be a positive number and select a maximal set of points $P_1, \dots, P_b$ on E, each pair of which are separated by a distance not less than r . Construct spheres of radius r with centers on $P_1, \dots, P_b$. The spheres define a neighborhood N' of edge E. Consider, also, the neighborhood N of E which can be formed by constructing spheres of radius $2r$ and centers on $P_1, \dots, P_b$. Observe that there is a number e independent of r and such that any sphere of radius $4r$ contains at most e of the points $P_1, \dots, P_b$. Let $H_j(x, r) \equiv h\left(\frac{d_j(x)}{r}\right)$, where $h(t)$ is the function defined in Section 3, and $d_j(x)$ is the distance of point (x) from the point P_j . Setting $H(x, r) \equiv H_1 H_2 \dots H_b$ it is seen that outside the neighborhood N of E, $H = 1$ and $H_{x_a} = 0$, while inside the neighborhood N' of E, $H = 0$ and $H_{x_a} = 0$. Moreover, since at most e of the functions H_j are different from 1, it follows that the absolute value of H_{x_a} is not greater than Ge/r , where G is an upper bound of $h'(t)$. Consider now an admissible function $\eta(x)$ and let $\mathcal{J} = H\eta$. Since $\mathcal{J} = 0$ in the neighborhood N' of E, it can be shown that equation (2.1) holds for $\mathcal{J}$ by an argument similar to that used in proof of Theorem 3.2, where the η used there is replaced by $\mathcal{J}$. Thus we have

$$\int_A (v_a \mathcal{J}_{x_a} + w \mathcal{J}) dA = \int_B \mathcal{J} v_a \lambda_a dB.$$

Replacing $\int$ by $H\eta$, we find that

$$\int_A H(v_\alpha \pi_\alpha + w\eta) + \int_A \eta v_\alpha H_{x_\alpha} dA = \int_B H\eta v_\alpha \lambda_\alpha dB.$$

In the limit as r tends to zero, this equation will become the equivalent of equation (2.1) if it can be shown that the second integral has zero as its limit. Let M be an upper bound of the values of ηv_α on A . Observe that the m -dimensional measure of AN cannot exceed $(q/r^s)cr^m$, where c is a constant. Since H_{x_α} is identically zero on $A - AN$, the absolute value of the second integral is not greater than

$$(Mm)(G/r)(q/r^s)cr^m = MmGcq \cdot r^{m-s-1}$$

which tends to zero with r , since $s < m - 1$ and the remaining terms are constants. This completes the proof of Theorem 5.1.

Theorem 5.2. The results described in Sections 3 and 4 hold if the class of admissible regions A is extended to include regions whose boundary B can be subdivided into a finite number of non-singular $(m - 1)$ -dimensional subspaces of class c' having the following property: The boundary of each $(m - 1)$ -dimensional subspace of B can be subdivided into a finite number of parts, each of which can be represented by a set of functions of the form

$$(5.1) \quad x_\alpha(t_1, \dots, t_{m-2}), \quad (t \text{ on } T; \alpha = 1, \dots, m)$$

which are Lipschitzian on T , the set T being bounded and closed.

It will be sufficient to prove that the intersections of the $(m - 1)$ -dimensional subspaces of B form a set E for which the hypotheses of Theorem 5.1 are satisfied. To do so we restrict ourselves to one piece of the set E given by equations of the form (5.1). This procedure is valid because if each piece of this type has the property described in Theorem 5.1, the whole set E has

this property, since E is composed of a finite number of such pieces. Consider, therefore, one of the subdivisions E' of E given by equations (5.1) and let K be a positive number such that

$$(5.2) \quad \|x_\alpha(t) - x_\alpha(t')\| < K \|t - t'\|$$

where $\|x\|$ is the positive square root of $x_1^2 + \dots + x_m^2$ and $\|t\|$ is the positive square root of $t_1^2 + \dots + t_{m-2}^2$. Let r be a positive number and let $P_1, \dots, P_b$ be a maximal set of points on the subset E' given by equations (5.1) and such that any two points are separated by a distance $d \geq r$. Let $T_1, \dots, T_b$ be the points in T which correspond respectively to the points $P_1, \dots, P_b$. By relation (5.2) the points $T_1, \dots, T_b$ are at a distance $d \geq r/K$ apart. The number of points of this type will be shown not to exceed a quantity q/r^{m-2} , where q is a constant independent of r . To prove this we can suppose that T is an $(m-2)$ -dimensional cube of side c . We now choose the greatest integer $h < cK/r$. Hence $r < cK/h \equiv r'$. It is to be observed that $r \geq r'/2$ since $2h \geq cK/r$. If we divide T into h^{m-2} equal subcubes of side c/h , the number of points in a maximal set in one of these subcubes, at a distance $d \geq r'/2K = c/2h$ apart is exceeded by a number q' which is independent of h . The number of points in T in a maximal set at a distance $d \geq r/K \geq r'/2K$, therefore does not exceed

$$q'h^{m-2} = q'(cK/r')^{m-2} < q'(cK/r)^{m-2} = q/r^{m-2}$$

where $q = q'(cK)^{m-2}$ is a constant independent of r . This completes the proof of Theorem 5.2.

As a special case of this theorem for three-dimensional space we have the following result.

Theorem 5.3. When region A is three-dimensional, the results of Sections 3 and 4 are valid for the case in which the

boundary B is composed of non-singular surfaces of class c' which intersect in a finite number of rectifiable curves.

This follows because the functions $x_\alpha(s)$ defining a rectifiable curve are Lipschitzian, when the parameter s is the arc length.

6. Extension of the class of functions v_α . In this section, we shall consider that the functions v_α are such that $A + B$ can be sub-divided into a finite number of parts on each of which v_α are continuous and such that each part is a region of the type described in Section 5. The following result will be proved.

Theorem 6.1. Equation (2.1) holds for every admissible function $\eta(x)$ if and only if the following two conditions hold: (1) at each point on an $(m - 1)$ -dimensional manifold R of discontinuity of v_α the function $v_\alpha \chi_\alpha$ is continuous along the normal to R , whose direction cosines are represented by χ_α ; (2) equation (4.1) holds for every m -dimensional cube A' which has its closure in A , its faces parallel to the coordinate hyperplanes, and contains no point of discontinuity of one of the functions v_α .

First we shall prove that criteria (1) and (2) described in the theorem are sufficient to insure that equation (2.1) holds for every admissible function $\eta(x)$. Let $A_1, \dots, A_c$ be the admissible subregions of A on which v_α are continuous and let $B_1, \dots, B_c$ be their boundaries. By criterion (2) and Theorem 4.1, the formula (2.1) holds for each A_i and hence for their sum. We thus obtain

$$\begin{aligned}
 6.1) \quad \int_A (v_\alpha \pi_\alpha + w\eta) dA &= \int_B \eta v_\alpha \chi_\alpha dB \\
 &+ \sum_{i,j} \int_{R_{ij}} \eta (\chi_\alpha^i v_\alpha^i + \chi_\alpha^j v_\alpha^j) dR,
 \end{aligned}$$

where the symbol R_{ij} represents the common boundary of the subregions A_i and A_j and the summation is taken over the A_i and A_j

which have a common boundary. The symbols χ_a^1 and χ_a^j represent respectively the direction cosines of the outer normals to A_1 and A_j . Since by criterion (1), the last integral has the value zero, we have that equation (2.1) holds for every admissible function $\eta(x)$.

On the other hand, if equation (2.1) holds for every admissible function $\eta(x)$ then it holds for each subregion A_1 as described above, and hence equation (6.1) is satisfied. Since equation (2.1) holds for $A + B$, the last integral is zero for every admissible function $\eta(x)$, which is possible only if $\chi_a v_a$ is continuous along the normal to R . Thus we have shown that criterion (1) is necessary.

To prove the necessity of criterion (2) consider a cube $A' + B'$ as described in the theorem. Since equation (2.1) holds for every admissible function $\eta(x)$ on cube $A' + B'$, take $\eta = 1$. Thus it is seen that equation (4.1) holds and that criterion (2) is necessary. Furthermore, by using Corollary 1 of Theorem 4.1, it can be shown that equation (4.1) must hold for each admissible subregion A_1 on which the functions v_a are continuous.

7. Applications. In this section we shall prove a few theorems which are useful in mechanics and elsewhere. It is to be noted that the proofs hold for an m -dimensional region A with boundary B of the type described in Section 5.

Theorem 7.1. Suppose the equations

$$(7.1) \quad \int_{A_0} f(x) dA = \int_{B_0} u_{x_a}(x) \chi_a dB,$$

$$(7.2) \quad \int_{A_0} g(x) dA = \int_{B_0} v_{x_a}(x) \chi_a dB,$$

hold for every m -dimensional cube A_0 whose closure is in A and whose faces are parallel to the coordinate hyperplanes; where $f(x)$

and $g(x)$ are integrable in A and the functions $u(x)$, $v(x)$ are of class c' on $A + B$. Then the following equation holds

$$(7.3) \quad \int_A (ug - vf) dA = \int_B (uv_{x_\alpha} - vu_{x_\alpha}) \lambda_\alpha dB.$$

By Theorem 4.1, the hypothesis concerning equation (7.1) implies that the equation

$$(7.4) \quad \int_A (u_{x_\alpha} \eta_\alpha + f\eta) dA = \int_B \eta u_{x_\alpha} \lambda_\alpha dB$$

holds for any admissible function $\eta(x)$. Choosing $\eta = v$ in this equation we obtain

$$(7.5) \quad \int_A (u_{x_\alpha} v_{x_\alpha} + vf) dA = \int_B vu_{x_\alpha} \lambda_\alpha dB.$$

By a similar method, we find the equation

$$(7.6) \quad \int_A (v_{x_\alpha} u_{x_\alpha} + ug) dA = \int_B uv_{x_\alpha} \lambda_\alpha dB.$$

Now subtracting equation (7.5) from equation (7.6) the desired result is found.

As a consequence of Theorem 7.1, we have

Theorem 7.2. Suppose that $u(x)$ and $v(x)$ are of class c' on $A + B$ and that their second derivatives exist and are integrable on A . Then the following equation is satisfied

$$(7.7) \quad \int_A [u(\Delta v) - v(\Delta u)] dA = \int_B (u \frac{dv}{dn} - v \frac{du}{dn}) dB$$

where Δv is the Laplacian $\Delta v = v_{x_\alpha x_\alpha}$ and $\frac{dv}{dn}$ is the outer normal derivative.

By Theorem 4.2, it follows that equations (7.1) and (7.2) hold with $f = \Delta u$ and $g = \Delta v$, so that the desired result follows immediately from Theorem 7.1.

Theorem 7.3. Suppose that $v(x)$ is of class c' on $A + B$ and that its second derivatives exist and are integrable on A .

Then if $u(x)$ is an admissible function the following equation holds

$$(7.7) \quad \int_A (u_{x_\alpha} v_{x_\alpha} + u \Delta v) dA = \int_B u \frac{dv}{dn}.$$

By Theorem 4.2, equation (2.1) holds if we let $w = \Delta v$ and replace v_α by v_{x_α} . If η is then set equal to u the result is equation (7.7), which completes the proof of this theorem.

In order to find a further interesting result, consider a set of functions $R_{\alpha\gamma}(x)$, ($\alpha = 1, \dots, m$; $\gamma = 1, \dots, m-1$) and the symbol $|\frac{\partial}{\partial x_\alpha} R_{\alpha\gamma}|$ which will represent the sum $\frac{\partial v_\alpha}{\partial x_\alpha}$, where v_α is the cofactor of χ_α in the determinant $|\chi_\alpha R_{\alpha\gamma}|$. We are now in position to state the following theorem.

Theorem 7.4. If the functions $R_{\alpha\gamma}(x)$, ($\alpha = 1, \dots, m$; $\gamma = 1, \dots, m-1$), are of class c^1 on A , we have

$$(7.8) \quad \int_A |\frac{\partial}{\partial x_\alpha} R_{\alpha\gamma}| dA = \int_B |\chi_\alpha R_{\alpha\gamma}| dB.$$

This result may be proved immediately by use of Theorem 4.3. Let v_α be the cofactor of χ_α in the determinant $|\chi_\alpha R_{\alpha\gamma}|$ and let $w = |\frac{\partial}{\partial x_\alpha} R_{\alpha\gamma}|$, then $w = \frac{\partial}{\partial x_\alpha} v_\alpha$. Application of Theorem 4.3 completes the proof.

8. Further results. We now restrict the admissible functions $\eta(x)$ to be non-negative on $A + B$ and we shall consider the relations

$$(8.1) \quad \int_A (v_\alpha \pi_\alpha + w \eta) dA \geq \int_B \eta v_\alpha \chi_\alpha dB,$$

$$(8.2) \quad \int_A w dA \geq \int_B v_\alpha \chi_\alpha dB.$$

The theorems of Sections 3, 4 and 5 and their proofs hold if equations (2.1) and (4.1) are replaced respectively by relations (8.1) and (8.2). Results similar to these and their applications are discussed by Levin [XIV] for the two-dimensional case under

less restrictive hypotheses on the functions v_α . The following theorem is formed by making the above mentioned changes in Theorems 3.2, 4.1 and 4.2. The boundary is considered to be of the type described in Section 5. The proofs are similar to those of the theorems just mentioned.

Theorem 8.1. The inequality (8.1) holds for every non-negative admissible function $\eta(x)$ if and only if it holds for every non-negative admissible function $\eta(x)$ vanishing on a neighborhood of the boundary B. Moreover, the inequality (8.1) holds for these functions if and only if the inequality (8.2) is satisfied for every m-dimensional cube A' whose closure is in A and whose faces are parallel to the coordinate hyperplanes. If the functions v_α have the property described in Theorem 4.2, then the inequality (8.1) holds for every non-negative admissible function if and only if the relation $w \geq v_{\alpha x_\alpha}$ holds almost everywhere on A . Finally, the above results hold if all inequalities are reversed or if all inequalities are replaced by equalities.

9. Formulation of the problem of the calculus of variations for multiple integrals. In the remainder of this paper the symbol x will represent a point in m -dimensional space X , as before, and y will represent a point in n -dimensional space Y . The symbols A and B will have the same significance as before, the boundary B being of the type described in Section 5. The partial derivatives $\frac{\partial y_i}{\partial x_\alpha}$ will be represented by p_1^α . Here and in the remaining sections the index i will range from 1 to n and α from 1 to m . Let R be a region of XYP space. A function $y(x)$ will be said to be of class D' on $A + B$ in case it is such that $A + B$ can be subdivided into a finite number of admissible subregions on each of which the function $y(x)$ is of class c' . An admissible manifold will be one having all of its elements

interior to R and with defining functions $y_1(x)$ of class D^1 .

Our problem is then to find among all admissible manifolds

$$y = y(x), \quad (x \text{ in } A)$$

having a common boundary L , one which minimizes the m -tuple integral

$$(9.1) \quad I = \int_A f(x, y, p) dA$$

where f is of class c^n in the region R .

10. Necessary conditions associated with the first variation. Let E be an admissible manifold defined by $y_1(x)$ and consider a one-parameter family of admissible manifolds

$$y = y(x) + \epsilon \eta(x)$$

where $\eta(x)$ is an admissible manifold vanishing on B . If we replace $y(x)$ by $y(x) + \epsilon \eta(x)$ in the integral (8.1), the result is

$$I(\epsilon) = \int_A f[x, y(x) + \epsilon \eta(x), \bar{p}] dA$$

where $\bar{p}$ represents the partial derivatives $\frac{\partial}{\partial x_a} [y_1(x) + \epsilon \eta(x)]$. If E is a minimizing manifold, then for ϵ sufficiently small $y(x) + \epsilon \eta(x)$ is an admissible manifold and $I(\epsilon)$ must have a minimum for $\epsilon = 0$. Differentiation of $I(\epsilon)$ with respect to ϵ yields

$$(10.1) \quad I'(\epsilon) = \int_A (f_{y_1} \eta_1 + f_{p_1^a} \pi_1^a) dA$$

where the symbol π_1^a represents $\frac{\partial \eta_1}{\partial x_a}$. Evaluating $I'(\epsilon)$ for $\epsilon = 0$, we have

$$(10.2) \quad I'(0) = \int_A (f_{y_1} \eta_1 + f_{p_1^a} \pi_1^a) dA$$

which is called the first variation of the integral I along manifold E . A necessary condition for E to be a minimizing manifold

is that $I'(0) = 0$ for every admissible variation $\eta(x)$ vanishing on B . Theorem 6.1 can be used to establish the following result.

Theorem 10.1. When $y(x)$ defines a minimizing manifold E of class D' , the equation

$$(10.3) \quad \int_A (f_{y_1} \eta_1 + f_{p_1^\alpha} \eta_1^\alpha) dA = \int_B \eta_1 f_{p_1^\alpha} \lambda_\alpha dB$$

holds for every admissible variation $\eta(x)$. Moreover, equation (10.3) holds on E for every admissible variation $\eta(x)$ if and only if the following two conditions are satisfied: (1) at each point on an $(m-1)$ -dimensional manifold R of discontinuity of $f_{p_1^\alpha}$ the function $f_{p_1^\alpha} \lambda_\alpha$ is continuous along the normal to R , whose direction cosines are represented by λ_α ; (2) the equations

$$(10.4) \quad \int_{A_0} f_{y_1} dA = \int_{B_0} f_{p_1^\alpha} \lambda_\alpha dB$$

hold for every m -dimensional cube A_0 which has its closure in A , its faces parallel to the coordinate hyperplanes, and contains no point of discontinuity of one of the functions $f_{p_1^\alpha}$.

It is to be noted that by Corollary 1 of Theorem 4.1, criterion (2) implies that equations (10.4) are satisfied for every subregion $A_0 + B_0$ of A on which the functions $y(x)$ are of class c' . These are known as the Haar-Coral [VI] equations.

By use of Theorem 4.2, we can establish the following result.

Theorem 10.2. (Euler equations). On every portion of a minimizing manifold E where the functions $y_1(x)$ are of class c'' , we have

$$(10.5) \quad f_{y_1} - \frac{\partial}{\partial x_\alpha} f_{p_1^\alpha} = 0.$$

As a further result we have [cf. XIV]

Theorem 10.3. If an admissible manifold E defined by

functions $y_1(x)$ of class c' affords a minimum to I in the class of admissible manifolds defined by functions $Y_1(x)$ for which $Y_1(x) \geq y_1(x)$ on A , then the inequality

$$(10.6) \quad \int_A (f_{y_1} \eta_1 + f_{p_1} \pi_1^\alpha) dA \geq \int_B \eta_1 f_{p_1}^\alpha \lambda_\alpha dB$$

holds on E for all non-negative admissible variations η_1 . The inequality (10.6) holds on E for these variations if and only if one has

$$(10.7) \quad \int_{A'} f_{y_1} dA \geq \int_{B'} f_{p_1}^\alpha \lambda_\alpha dB$$

on every cube A' whose closure is in A and whose faces are parallel to the coordinate hyperplanes. If E minimizes I in the class of admissible manifolds defined by functions $Y_1(x)$ for which $Y_1(x) \leq y_1(x)$ on A , the inequalities in (10.6) and (10.7) are reversed.

This result follows from Theorem 8.1 since if E minimizes I in class of functions $Y_1(x)$ for which $Y_1(x) \geq y_1(x)$ on A , then the first variation of I for $Y_1(x)$ satisfies the relation $I' \geq 0$ for non-negative admissible variations $\eta_1(x)$ vanishing on B . If, on the other hand, E minimizes I in the class of admissible manifolds defined by functions $Y_1(x)$ for which $Y_1(x) \leq y_1(x)$ one has $I'(0) \leq 0$ for non-negative admissible variations vanishing on B , so that the inequalities (10.6) and (10.7) are reversed.

11. Further necessary conditions. In this section we find further necessary conditions by transforming the problem into parametric form. We shall prove the following result.

Theorem 11.1. Let E be a minimizing manifold defined by functions $y(x)$. On every admissible subregion $A_0 + B_0$ of A on which the functions $y(x)$ are of class c' , the following equations hold

$$(11.1) \quad \int_{A_0} f_{x_\alpha} dA = \int_{B_0} (f_{\alpha\beta} - p_1^\alpha f_{p_1^\beta}) \lambda_\beta dB, \quad (\beta = 1, \dots, m).$$

Moreover, at each point P on a manifold R of discontinuity of the functions $f_{p_1^\beta}$ the function

$$(11.2) \quad (f_{\alpha\beta} - p_1^\alpha f_{p_1^\beta}) \lambda_\beta$$

is continuous along the normal to R, where λ_β represent the direction cosines of the normal.

In order to prove this result, we apply the transformation

$$(11.3) \quad x_\alpha = x_\alpha(t_1, \dots, t_m)$$

to the integral I. Then we have $y(x) = y(x(t)) = Y(t)$ and after differentiating, the result is

$$(11.4) \quad r_1^\beta = p_1^\alpha x_{\alpha t_\beta}$$

where the notation r_1^β represents Y_{1t_β} . After solving for p_1^α , the result is

$$(11.5) \quad p_1^\alpha = \frac{\Delta_{1\alpha}}{\Delta}$$

where $\Delta \equiv |x_{\alpha t_\beta}|$ and $\Delta_{1\alpha}$ is the determinant obtained from Δ by replacing the α -th row by r_1^β . These notations were used by Powell [XX, pp. 53-4]. The integral I then takes the form

$$(11.6) \quad \int_T F(x, y, q, r) dT$$

where $F(x, y, q, r) = f[x(t), Y(t), \frac{\Delta}{\Delta} \Delta, q_\alpha^\beta \equiv (x_{\alpha t_\beta})$, and T is the region in t-space corresponding to region A_0 by transformation (11.3). This is a problem of the same type as the original except that now the independent variable is t and we have $m + n$ dependent variables $x(t)$ and $Y(t)$. By application of Theorem 10.1 to this problem, we have

$$(11.7) \quad \int_T F_{x_\alpha} dT = \int_S F_{q_\alpha}^\beta L_\beta dS$$

where S is the boundary of T , and L_β are direction cosines of the outer normal to S .

The partial derivatives $F_{q_\alpha}^\beta$ were calculated by Powell [XX, pp. 54-5] and were found to be

$$(11.8) \quad \begin{aligned} F_{q_\alpha}^\alpha &= f - p_1^\alpha f_{p_1^\alpha} & (\alpha \text{ not summed}) \\ F_{q_\alpha}^\beta &= -p_1^\alpha f_{p_1^\beta} & (\alpha \neq \beta). \end{aligned}$$

These results may be stated in the form

$$(11.9) \quad F_{q_\alpha}^\beta = f \delta_{\alpha\beta} - p_1^\alpha f_{p_1^\beta}.$$

Thus, it is seen that when the transformation $t_\alpha = x_\alpha$ is applied to integral (11.6) the result is

$$(11.10) \quad \int_{A_0} f_{x_\alpha} dA = \int_{B_0} (f \delta_{\alpha\beta} - p_1^\alpha f_{p_1^\beta}) \chi_\beta dB.$$

This completes the proof of Theorem 11.1.

We have, also, the further result

Theorem 11.2. On every portion of a manifold E where the functions $y_1(x)$ are of class c'' , the following identity holds.

$$(11.11) \quad f_{x_\alpha} - \frac{\partial}{\partial x_\beta} (f \delta_{\alpha\beta} - p_1^\alpha f_{p_1^\beta}) + p_1^\alpha (f_{y_1} - \frac{\partial}{\partial x_\beta} f_{p_1^\beta}) \equiv 0.$$

From this result it follows that equations (11.1) are a consequence of equations (10.4) or (10.5) for a minimizing manifold E of class c'' .

12. Necessary conditions associated with the second variation. Throughout this section we shall consider a minimizing manifold E whose defining functions $y_1(x)$ are of class c' . If we differentiate equation (10.1) with respect to ϵ and evaluate at $\epsilon = 0$ we have the second variation

$$(12.1) \quad I''(0) = \int_A 2\Omega(x, \eta, \pi) dA$$

of the integral I along manifold E ; where $\eta = (\frac{\partial \eta_i}{\partial x_a})$ and 2Ω denotes the quadratic form $f_{y_1 y_k} \eta_1 \eta_k + 2f_{y_1 p_1^a} \eta_1 \pi_k^a + f_{p_1^a p_k^a} \pi_1^a \pi_k^a$, ($k = 1, \dots, n$; $a = 1, \dots, m$).

For any minimizing manifold $y(x)$ the relations $I'(0) = 0$ and $I''(0) \geq 0$ must hold. By Euler's theorem on homogeneous functions, we have

$$(12.2) \quad 2\Omega = \Omega \pi_1^a \pi_1^a + \Omega \eta_1 \eta_1.$$

By Legendre's condition* in the stronger form is meant that along a minimizing manifold the quadratic form

$$(12.3) \quad f_{p_1^a p_1^a} w_{a1}^1 w_{a1}^1$$

is positive for all matrices w_{a1}^1 of rank one.

We are now in a position to prove the following result.

Theorem 12.1. (Analogue of Jacobi's condition). Let $y = y(x)$ be a minimizing manifold E of class c' on which Legendre's condition in the stronger form is satisfied. Furthermore, let A_0 be an admissible subregion of A with boundary B_0 . Then there can be no solution $\eta_1 = u_1(x)$ of the n equations

$$(12.4) \quad \int_A \Omega \eta_1 dA = \int_{B_0} \Omega \pi_1^a \pi_1^a dB, \quad (i = 1, \dots, n)$$

for all subregions A' of A_0 and such that the functions $u_1(x)$ are of class c' on $A_0 + B_0$, vanish on B_0 , and have first partial derivatives not all identically zero at the points of B_0 in A .

In order to prove this theorem, let us suppose that the $u_1(x)$ satisfy the n equations (12.4), being of class c' on $A_0 + B_0$,

*For proofs of this necessary condition see the papers V, VII, IX and X at the end of this paper.

vanishing on B_0 and such that their first partial derivatives are not all zero at a point P of B_0 in A . A portion E of B_0 in a neighborhood of P can be represented by the equations

$$(12.5) \quad x_\alpha = g_\alpha(t_1, \dots, t_{m-1}), \quad \sum x_\alpha^2 \neq 0,$$

where λ_α represents the direction cosines of the normal to B_0 , and the functions g are of class C^1 .

Define the variation

$$(12.6) \quad \begin{aligned} \eta_i(x) &\equiv u_i(x) && \text{on } A_0 + B_0 \\ &\equiv 0 && \text{outside } A_0 + B_0. \end{aligned}$$

Making use of relation (12.2) we find that the second variation for $\eta(x)$ is

$$(12.7) \quad I''(0) = \int_A 2\Omega(x, \eta, \pi) dA = \int_A (\Omega_{\pi_1^a} \pi_1^a + \Omega_{\eta_1} \eta_1) dA.$$

Let us now consider the accessory problem of minimizing the integral (12.7) in the class of admissible manifolds $\eta(x)$. We have, by Theorems 6.1 and 3.2 that

$$(12.8) \quad I''(0) = \int_{A_0} (\Omega_{\pi_1^a} \pi_1^a + \Omega_{\eta_1} \eta_1) dA = 0$$

since η vanishes on B . Thus $I''(0)$ takes its minimum value for the variation $\eta(x)$. Therefore, $\eta_i(x)$ must satisfy the edge conditions near P , since these functions constitute a solution of class D^1 for the problem of minimizing $I''(0)$. Thus, we have

$$\lambda_\alpha \Omega_{\pi_1^a} \Big|_1 = \lambda_\alpha \Omega_{\pi_1^a} \Big|_2$$

where the numbers 1 and 2 indicate that the limiting values are approached along the normal, first from one side of the edge E and then from the other side. The right members of these equations vanish, leaving

$$(12.9) \quad \lambda_\alpha \Omega \gamma_1^\alpha (x, u, w) \Big|_1^1 = 0$$

on E , where $w \equiv \frac{\partial u_k}{\partial x_\alpha}$.

In addition to equations (12.9) we have at P the equations

$$(12.10) \quad du_1 = \frac{\partial u_1}{\partial x_\alpha} dx_\alpha = \frac{\partial u_1}{\partial x_\alpha} \frac{\partial g_\alpha}{\partial t_\gamma} dt_\gamma = 0, \quad (\gamma = 1, \dots, m-1).$$

Since the differentials dt_γ are arbitrary, equation (12.10) yields the equations

$$(12.11) \quad \frac{\partial u_1}{\partial x_\alpha} \frac{\partial g_\alpha}{\partial t_\gamma} = 0, \quad \text{or} \quad w_1^\alpha g_\alpha^\gamma = 0.$$

Leaving i fixed and solving for w_1^α gives

$$w_1^\alpha = \lambda_1 \lambda_\alpha.$$

Multiply through equations (12.9) by λ_1 and the result is

$$\begin{aligned} \lambda_1 \lambda_\alpha \Omega \gamma_1^\alpha &= \Omega \gamma_1^\alpha (\lambda_1 \lambda_\alpha) = \Omega \gamma_1^\alpha w_1^\alpha = \gamma_k^\beta f_{p_1}^\alpha p_k^\beta w_1^\alpha \\ &= u_k^\beta f_{p_1}^\alpha p_1^\beta \lambda_1 \lambda_\alpha = f_{p_1}^\alpha p_k^\beta \lambda_1 \lambda_\alpha \lambda_k \lambda_\beta = 0. \end{aligned}$$

This contradicts the Legendre condition, since the matrix w_1^α is of rank one. Then the admissible variation (12.6) cannot give $I''(0)$ a minimum, which contradicts the hypothesis that $y(x)$ is a minimizing manifold. This completes the proof of Theorem 12.1.

As a final result we have the following generalization of Theorem 7.2.

Theorem 12.2. If ξ_1 and η_1 are admissible variations of class c'' then the equations

$$(12.12) \quad \int_{A_0} [\xi_1 J_1(\eta) - \eta_1 J(\xi)] dA = \int_{B_0} (\eta_1 \Omega_{k_1}^\alpha - \xi_1 \Omega_{\eta_1}^\alpha) \lambda_\alpha dB$$

hold for every admissible subregion A_0 of A , where $J_1(\eta)$ are the accessory equations

$$(12.13) \quad J_1(\eta) = \Omega \eta_1 - \frac{\partial}{\partial x_\alpha} \Omega \eta_1^\alpha.$$

The symbol k_1^α represents ξ_{ix_α} , the arguments in $\Omega_{k_1^\alpha}$ and $\Omega_{\pi_1^\alpha}$ are (x, ξ, k) and (x, η, π) , respectively, and λ_α are the direction cosines of the outer normal to B_0 .

In order to prove this result, let us form the identity

$$(12.13) \quad \int_{A_0} \xi_{iJ_1}(\eta) dA = \int_{A_0} (\xi_{i1} \Omega_{\eta_1} + \xi_{i1} \Omega_{\pi_1^\alpha}) dA - \int_{A_0} \frac{\partial}{\partial x_\alpha} (\xi_{i1} \Omega_{\pi_1^\alpha}) dA$$

which, by Green's theorem, takes the form

$$(12.14) \quad \int_{A_0} \xi_{iJ}(\eta) dA = \int_{A_0} (\xi_{i1} \Omega_{\eta_1} + \xi_{i1} \Omega_{\pi_1^\alpha}) dA - \int_{B_0} \xi_{i1} \Omega_{\pi_1^\alpha} \lambda_\alpha dB.$$

Similarly, we can form the equation

$$(12.15) \quad \int_{A_0} \eta_{iJ_1}(\xi) dA = \int_{A_0} (\eta_{i1} \Omega_{\xi_1} + \eta_{i1} \Omega_{k_1^\alpha}) dA - \int_{B_0} \eta_{i1} \Omega_{k_1^\alpha} \lambda_\alpha dB.$$

Subtracting equation (12.15) from (12.14) and making use of relation (12.2), the result is equation (12.12). This completes the proof of Theorem 12.2.

BIBLIOGRAPHY

- I. Bray, H. E., A Green's theorem in terms of Lebesgue integrals, Annals of Mathematics, vol. 21 (1920), pp. 141-56.
- II. Bray, H. E., Green's lemma, Annals of Mathematics, vol. 26 (1924-5), pp. 278-86.
- III. Brouwer, L. E. J., Meerdimensionale Vectordistribues, Amsk. Ak. Versl., vol. 15 (1906), p. 14.
- IV. Brouwer, L. E. J., Opmerking over meervoudige integralen, vol. 28 (1919), 116-20.
- V. Caratheodory, C., Ueber die Variationsrechnung bei Mehrfachen Integralen, Acta Litt. ac Scient., vol. 4 (1929), p. 193.
- VI. Coral, M., On the necessary conditions for a minimum of a double integral, Duke Mathematical Journal, vol. 3 (1937).
- VII. Graves, L. M., The Weierstrass condition for multiple integral variation problems, Duke Mathematical Journal, vol. 5 (1939), pp. 656-660.
- VIII. Grosz, A proof of Green's theorem for rectifiable boundary, Monatshifte fur Mathematik und Physik, vol. 27 (1916), p. 70.
- IX. Hadamard, J., Sur une question de calcul des variations, Bull. de la Soc. Math. de France, vol. 30 (1902), p. 253.
- X. Hadamard, J., Sur quelques questions de calcul des variations, Bull. de la Soc. Math. de France, vol. 33 (1905), p. 73.
- XI. Heilbronn, Zu dem Integralsatz von Cauchy, Mathematische Zeitschrift, vol. 37 (1933), pp. 37-38.
- XII. Hestenes, M. R., Analogue of Green's theorem in the calculus of variations, Duke Mathematical Journal, vol. 8 (1941)..
- XIII. Huke, A., An historical and critical study of the fundamental lemma of the calculus of variations, Dissertation, Chicago, 1930.
- XIV. Levin, J. H., Minima of double integrals with respect to unilateral variations, and applications to subharmonic functions, Dissertation, Chicago, 1940.
- XV. Mason, M., Beweis eines Lemmas der variationsrechnung, Math. Annalen, vol. 61 (1905), p. 450.
- XVI. McShane, E., Weierstrass condition in the multiple integral problem, Annals of Math., vol. 32 (1931), p. 578.

- XVII. Menger, K., On Green's formula, Proc. of National Academy of Sciences, vol. 26, no. 11, pp. 660-64.
- XVIII. Picone, M., Sul teorema di Green nel piano e nello spazio.
- XIX. Pollard, S., On the conditions for Cauchy's Theorem, Proceedings of London Math. Soc., vol. 21 (1922).
- XX. Powell, J. E., Edge conditions for multiple integrals in the calculus of variations, Thesis, Contributions to the Calculus of Variations 1931-2, The University of Chicago Press.
- XXI. Raab, A. W., Jacobi's condition for multiple integral problems of the calculus of variations, Thesis, Contributions to the Calculus of Variations 1931-2, The University of Chicago Press.
- XXII. Reid, W. T., The Jacobi condition for the double integral problem of the calculus of variations, Duke Mathematical Journal, vol. 5 (1939), pp. 856-870.
- XXIII. Reid, W. T., Green's lemma and related results, American Journal of Mathematics, vol. 8 (1941).
- XXIV. Van Vleck, E. B., An extension of Green's lemma to the case of a rectifiable boundary, Annals of Math., vol. 22 (1921), pp. 226-237.

VITA

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